Aerodynamics of a flying ya

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I. The Chicken from Minsk

In a Soviet-era collection of physics exercises called "The Chicken from Minsk" by Yuri Chernyak and Robert Rose [1], one problem recalls air to air rockets launched from the tail end of a bomber in flight. A rocket with fins near its tail (like fletch of a ya) is mechanically catapulted, and immediately "sees itself" oriented tail-first in an enormous head wind. Soon after, the rocket's motor ignites... .

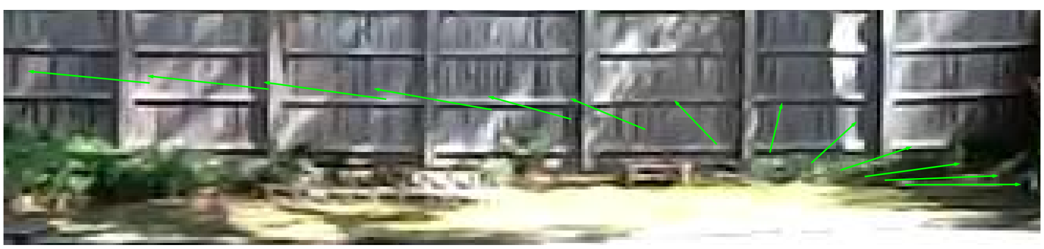


Figure 1: Chicken from Minsk experiment-archery version

Here is an archery version of this "experiment:" The positions of yanone (point) and hazu (nock) are exchanged on a 104cm long 2015 aluminum ya with standard fletching. The ya is launched fletch first towards a backstop 14m downrange. The ya flight is videoed at 240 frames per second. The ya is rather faint and blurry but still discernible on the video. Every sixth frame is imported into a graphics program. For each frame, the ya is traced by a line segment, and a graphical arrowhead indicates the end with the *hazu*. Figure 1 displays the complete sequence of oriented line segments. It amounts to a *time series* of ya configurations. The time increment is $\frac{6}{240}\ s = \frac{1}{40}\ s$. The ya configurations during the flipping appear foreshortened because the plane of the flipping is not vertical. Also notice that the flipping greatly decreases the speed of the ya. The drag on the ya is increased when its axis is perpendicular to the direction of flight.

The intuition is simple, from the ya's point of view: The ya, like the rocket, "sees itself" fletch first in a headwind. In figure 1 we see that the ya initially advances by its whole length in one time increment, so the headwind has an initial speed of $40m/s$. With any small mis-alignment between the ya's axis and the direction of the headwind, the headwind catches the fletch and flips the ya, which continues down range hazu first. This amusing experiment is a most transparent realization of underlying principles which we engage in the next section.

A second experiment, less overtly ridiculous, is informed by those same principles. Figure 2 is a photograph of a fletched ya, $102cm$ long, fitted with a "watakuri" arrowhead improvised in my workshop. This ya is shot from a

Figure 2: Ya with "watakuri" arrowhead

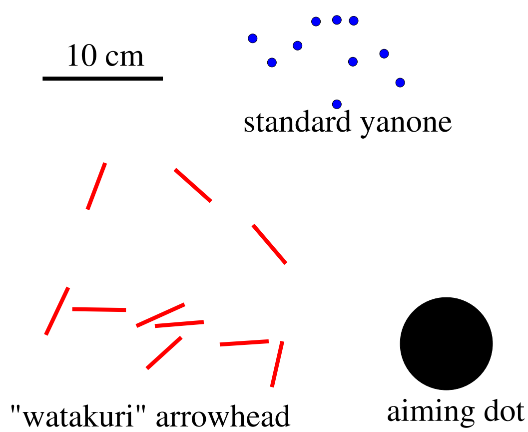


Figure 3: "Watakuri" arrowhead degrades grouping

$15kg$ yumi at a backstop $10m$ downrange. At Kai, the tip of the ya kisses an aiming dot at 6 o'clock. Ten shots are done with the "watakuri" arrowhead.

Ten more are done with the watakuri arrowhead replaced by a standard yanone. Figure 3 is the record of the impacts: A photograph of the backstop with the arrow holes is imported into the graphics program. The slashes produced by watakuri shots are traced by the red slits, the holes produced by normal yanone, by blue dots. The grouping of the watakuri shots is distinctly poorer. Apparently, the watakuri arrowhead with its broad, wind-catching shape acts like "fletching on the wrong end," a proximate cause of unstable and erratic arrow flight.

This essay addresses underlying principles of ya aerodynamics beyond simple intuition. Principles in hand, we may critically evaluate various preferences for preparing ya. For instance: What are the relative merits of forward versus back weighted ya? Does it really matter? How much fletching do we really need for good ya flight? Are traditional ya under-fletched or over-fletched?

II. Forward motion and alignment interact

Figure 4 is a snapshot of a flying ya. For a simplified first look, we assume that the ya at all times is confined to a vertical plane. The white dot

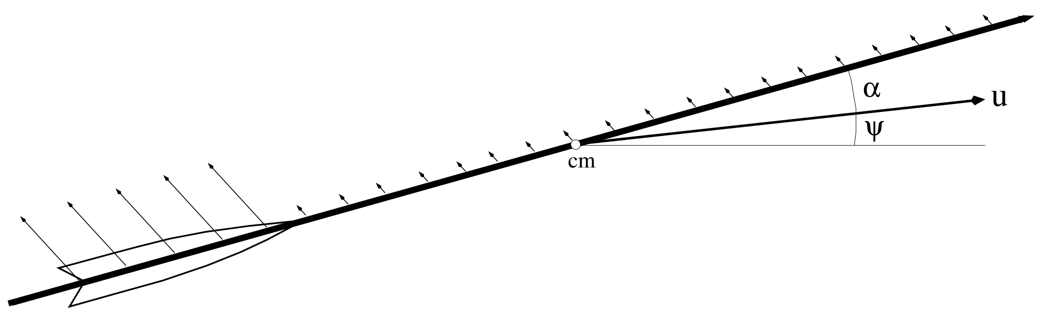


Figure 4: Force per unit length along a flying ya

marks its center of mass. The arrow labeled **u** indicates the magnitude and direction of the center of mass velocity. The angle α of the shaft relative to its direction of motion is called its *angle of attack*, as depicted in figure 4. The angle between the direction of velocity **u** and the horizontal is denoted by ψ. We'll call it the *angle of motion*. There is an interactive dance between the two angles α and ψ induced by aerodynamic forces, the role of gravity being small. The small arrows above the ya shaft qualitatively indicate the aerodynamic force per unit length acting along the shaft. The projection of a force arrow onto the direction opposite **u** is called *drag*. The projection in the orthogonal direction counterclockwise relative to $\mathbf{u}$ is called *lift*. Given the angle of attack as shown, the lift ought to be *upwards*. The force arrows along the. fletched section should be longer than the rest. The arrows drawn in figure 4 reflect this "common sense." As is well known in mechanics, the center of mass moves as if it were a point particle subject to the collective actions of all the forces along the entire length of ya. Due to the "swept back" orientations of the force arrows, the collective force slows down the arrow. Simultaneously, the upward lift increases the angle of motion ψ. In addition, the aerodynamic forces induce a pivoting of the arrow shaft about its center of mass. The extra large force arrows along the fletched section *behind* the center of mass exert a *clockwise* torque that tends to decrease the "point up" attitude of the ya relative to its velocity $\mathbf{u}$. If the ya is "point down" relative to its direction of motion, there is a *counterclockwise* torque that tends to decrease the "point down" attitude. In summary, fletching behind the center of mass induces a *restoring torque* that tries to align the ya with its velocity. Fletching front of center induces a *destabilizing* torque so the tendency is to pivot away from alignment. Figure 5 visualizes the physical situation of the Chicken from Minsk rocket "experiment." A watakuri arrowhead produces the same kind of "flipping" torque. In practice, the flipping induced by a bladed arrowhead is countermanded by sufficiently large fletching near the nock. Apparently, the fletching of the ya in figure 2 is not quite big enough to suppress the destabilizing tendencies of the watakuri arrowhead. In summary,

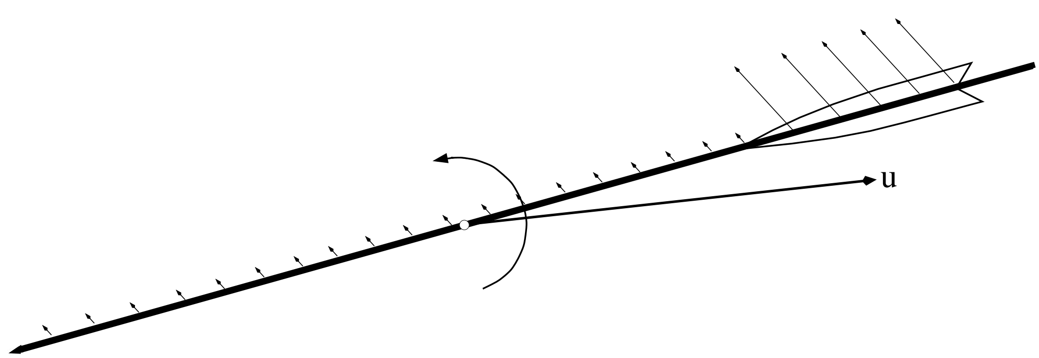


Figure 5: Lift generated torque induces flipping in the Chicken from Minsk "experiment"

we've outlined the *qualitative* story of how the angles of attack and motion are in a "dance" with each changing the other.

Can we do a *quantitative* analysis? Yes, but *not* by simple use of first principles. The onrushing headwind seen by the ya is disrupted by the ya itself. The disruptions are far too complicated for simple analysis of aerodynamic forces. Aeronautical engineers traditionally rely on wind tunnel tests. In particular, Takeshi Miyazaki et al [2] and Julio Ortiz [3] *measure* the aerodynamic forces and torques on an Olympic arrow in a wind tunnel at different angles of attack. With this information, the interactive dynamics between the angles of attack and motion becomes clear. In the appendix, we review the measurements and the simple theory enabled by them.

Figure 6 is a qualitative movie of a "porpoising" ya based on this analysis. The angle of attack cycles through tip down and tip up attitudes. The angle of motion undergoes a collateral cycle, which induces the elevation of the ya to cycle as well. Notice that the ya is tip down at its highest elevation and tip up at the lowest. The difference between the highest and lowest elevations has been exaggerated for clarity. For the Olympic arrow tested in the wind tunnel, we can crunch the numbers to see that the elevation changes induced by the oscillating angle of attack are on the order of a few millimeters. Quite insignificant.

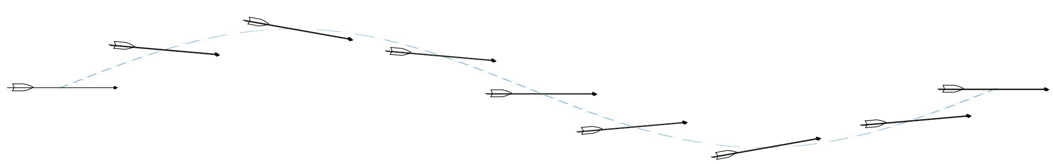


Figure 6: A movie of "porpoising."

III. Less is more

The questions (i) Forward versus back weighted ya? and (ii) What size of fletch? are informed by the aerodynamics. We know that fletching behind the ya's balance point stabilizes straight, point forward ya flight. Presumably, the farther behind, the better. The Olympic arrow tested by Takeshi Miyazaki et al has very small fletch in comparison to Kyudo ya: Length on the order of $5cm$ and height on the order of $1cm$. Nevertheless, the total lift force induced by a nonzero angle of attack acts as if it is concentrated just in front of the fletching at a point called the *lift center*. The arrow flight remains stable, so long as the the center of mass remains well forward of the lift center.

Now, let's look at Kyudo ya: Traditional take-ya have very light yanone so they balance very nearly at their midpoints. Traditional fletch are on the

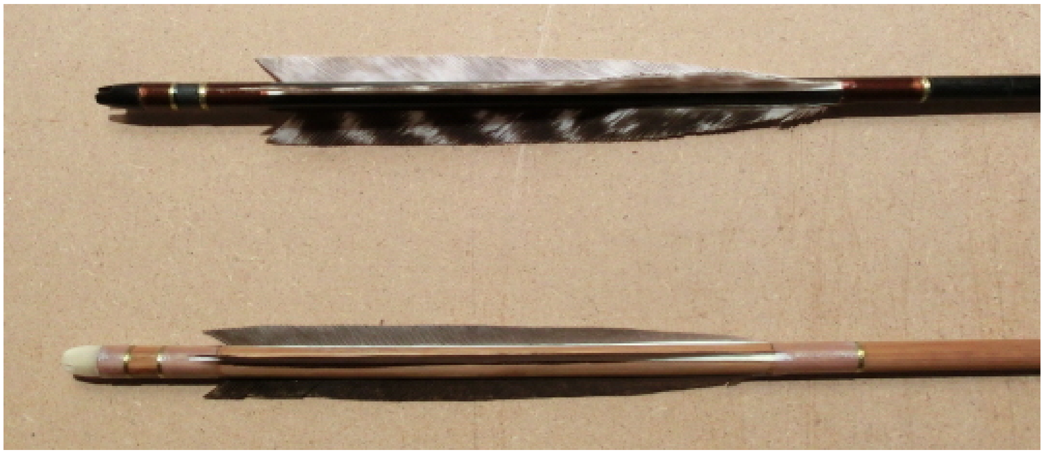


Figure 7: "Standard" and "low profile" fletching

order of 15cm long and between 1.0cm and 1.5cm high. Yes, the ya is longer than the Olympic arrow, but even so, ya fletch are *proportionally* longer and higher. The center of lift remains near the fletchings. Forward or back weighting of the ya which moves the center of mass a few centimeters doesn't change the forward displacement of the center of mass from the lift center by much. You can add enough weight to the yanone so the balance point moves significantly forward, but who wants to turn a 30gm ya into a 40gm ya? A center-balanced ya with *no* weight added is a reasonable choice after all. Is it possible that the fletch can be reduced a lot, but ample stability remains? Figure 7 is a photograph comparing "standard" and "low profile" fletching. The ya with the low profile fletching is 104cm long, weighs 32.5gm, and its stiffness is 17% greater than a 2015 aluminum shaft. The fletch is 14.4cm long but *only* 6mm *high*. It is shot from a 15kg yumi. I expected decent ya flight. I got close to perfect arrow flight with no discernible "porpoising" or "fishtailing" (fishtailing is like porpoising but the pivoting is about a vertical instead of horizontal axis.). I'm not going to say that such low profile fletching "caused" good ya flight. Most likely, a ya with appropriate stiffness relative to the strength of the yumi and the archer's yazuka flies very well with reduced fletching which still provides ample stability.

Appendix. A simple model of aerodynamic alignment aided by wind tunnel tests

Figure 8 is a snapshot of a ya in flight. We assume that the ya is confined to the vertical x, z plane. The degrees of freedom are $\mathbf{x}$, the ya's center of mass position, and the angle θ of the ya's axis with respect to the horizontal. The ya is subject to forces along its length. Let s, $0 < s < l$ be the distance along the ya from its nock end (tail). Let $\boldsymbol{\phi}(s,t)$ denote the force per unit length as a function of s and time t. For now, we assume it is given. The *Lagrangian* for the dynamics of $\mathbf{x} = \mathbf{x}(t)$ and $\theta = \theta(t)$ is

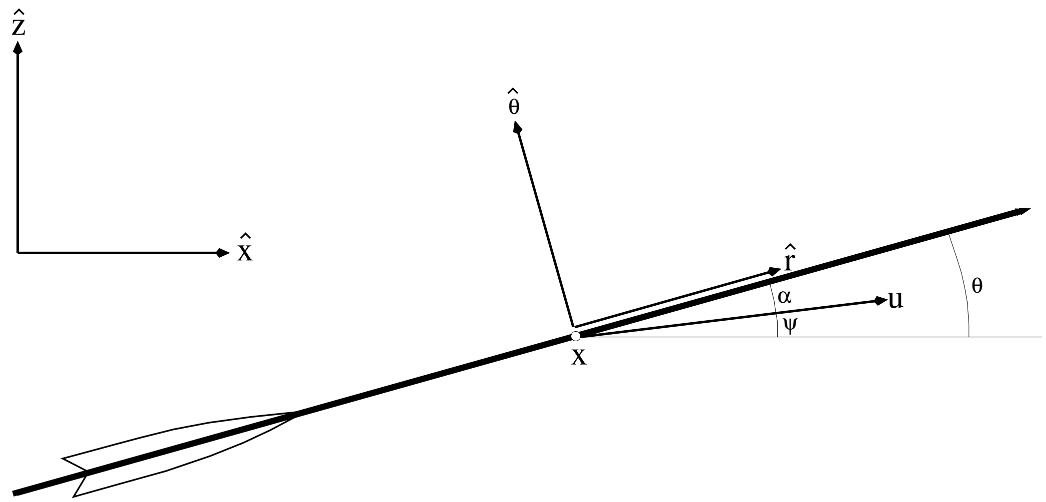


Figure 8: The arrow's degrees of freedom

$$L = \frac{1}{2}m|\dot{\mathbf{x}}|^2 + \frac{1}{2}I\dot{\theta}^2 + \int_0^l \boldsymbol{\phi} \cdot (\mathbf{x} + (s - S)\hat{\mathbf{r}})ds. \tag{1}$$

Here, m is the mass of the ya. I is its moment of inertia about the perpendicular axis through the center of mass. $\hat{\mathbf{r}}$ denotes the unit vector pointing from the ya's tail to its head. S denotes the distance of the ya's center of mass from the nock end. Notice that $\mathbf{y}(t) := \mathbf{x}(t) + (s - S)\hat{\mathbf{r}}$ is the trajectory of a material point with distance s from the nock end. The Lagrangian equations of motion are

$$m\dot{\mathbf{u}}(t) = \mathbf{f}(t) := \int_0^l \boldsymbol{\phi}(s,t)ds, \tag{2}$$

$$I\ddot{\theta}(t) = M(t) := \int_0^l (s - S)\hat{\theta} \cdot \boldsymbol{\phi}(s,t)ds. \tag{3}$$

$\mathbf{u} := \dot{\mathbf{x}}$ is the ya's velocity. We recognize $\mathbf{f}$ as the total force acting on the ya. The content of (2) is that the center of mass moves like a point particle subject to this total force. In (3), $\hat{\theta}$ is the unit vector normal to the shaft, obtained by rotating $\hat{\mathbf{r}}$ by $\frac{\pi}{2}$ counterclockwise radians. We recognize M as the *torque* applied about the center of mass.

The alignment of the ya with its direction of flight is due to aerodynamic forces, gravity playing a small role. We assume that the net force $\mathbf{f}$ and the torque M have aerodynamic origins. The aerodynamics is complicated. There are no simple formulas for force and torque based on first principles. Instead, the first principles we do know provide a framework for *measurements*. These have been done for arrow aerodynamics by Takeshi Miyazaki et al in [1] and Julio Ortiz et al in [2].

The framework begins with scale covariance: The aerodynamic force depends on dimensional quantities, most strongly on air density ρ, arrow speed $u := |\mathbf{u}|$, and the arrow's cross sectional area σ. The only combination with the units of force is $\rho\sigma u^2$. Hence, the force is represented by

$$\mathbf{f} = \frac{1}{2}\mathbf{c}\rho\sigma u^2, \tag{4}$$

where $\mathbf{c}$ is a dimensionless two-vector. The component c_d in the direction opposite $\mathbf{u}$ is called the *drag coefficient*. The component c_l in the orthogonal direction obtained by counterclockwise rotation of $\mathbf{u}$ is called the *lift coefficient*. We have

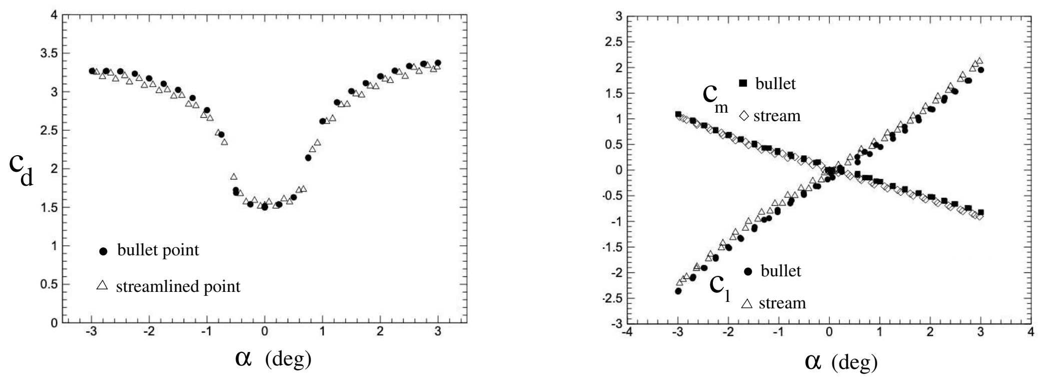
$$\mathbf{c} = -c_d\hat{\mathbf{u}} + c_l J\hat{\mathbf{u}}, \tag{5}$$

where $\hat{\mathbf{u}}$ is the unit vector in the $\mathbf{u}$ direction, and J is rotation by $\frac{\pi}{2}$ counterclockwise radians. The aerodynamic torque M is represented by

$$M = \frac{1}{2}c_m\rho\sigma l u^2. \tag{6}$$

Here, $\rho\sigma l u^2$ carries the units of torque (force times length), so c_m is another dimensionless quantity, called the *pitching moment coefficient*.

For a given ya flying at a given speed, the drag, lift and pitching coefficients depend on the *angle of attack* α. Referring to figure 8, the angle of attack is the angle of the ya axis relative to its direction of motion. [2] and [3] report their measurements of c_d, c_l, c_m as functions of α for typical arrows used in Olympic competition. In their elegant test setup, small cylindrical magnets are inserted in the shaft, which allows them to "levitate" the arrow in the wind tunnel by applying a magnetic field. By adjusting the magnetic field,, they can induce a given angle of attack, and deduce the force and



(a) (b)

Figure 9: The Easton ACE arrow is 62.5cm long, with 5.25mm mean diameter. Two points are used: A standard issue "bullet" point and a specially machined "streamlined" point. It is evident that the type of point had little influence on test results. The fletching is "spin wing vanes" typically used by Olympic archers. The total mass of the assembled arrow with point and fletchings is 14.3gm. The air speed in the wind tunnel is 31.4m/sec. This is close to half of the typical arrow speed in Olympic competition.

torque required to maintain it. Figure 9 reproduces their graphs of c_d, c_l, c_m versus α in [1]. Notice that the graphs of lift and pitching coefficients are asymptotically linear as $\alpha \to 0$. We have

$$c_l(\alpha) \sim a\alpha, \quad c_m(\alpha) \sim -b\alpha. \tag{7}$$

From the graphs of figure 9b, we discern values of the proportionality constants (converted to inverse radians):

$$a \approx 35.8\, rad^{-1}, \quad b \approx 17.9\, rad^{-1}. \tag{8}$$

What do the *negative* slopes of the pitching coefficient graphs mean? Figure 10 depicts the ya with a small angle of attack. The torque M is due mainly to the lifting force. The *lift center* is the point on the shaft so that the total lift force concentrated there gives the observed torque. Let ΔS be the displacement of this lift center from the center of mass, negative if the lift center is *behind* the center of mass. The torque reckoned in this way is

$$\frac{1}{2}\rho\sigma u^2 \Delta S a. \tag{9}$$

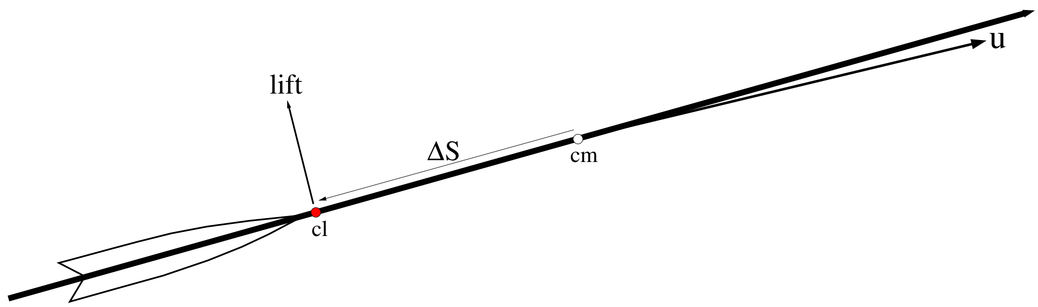


Figure 10: Negative torque due to center of lift behind center of mass

The torque according to (6), (7) is

$$-\frac{1}{2}\rho\sigma u^2 lb. \tag{10}$$

Equality of (9) and (10) yields

$$\Delta S = -\frac{b}{a}l \approx -\left(\frac{17.9}{35.8}\right)(62.5cm) \approx -31cm \tag{11}$$

for the arrow with length $62.5cm$. Indeed Takeshi Miyazaki et al report that the center of lift is "about $30cm$ behind the center of mass."

Taking the dependences of drag, lift and pitching coefficients upon the angle of attack α as given, equations (2), (3) become a closed dynamical system for $\mathbf{u}$ and θ. We may represent $\mathbf{u}$ by its magnitude (speed) u and its angle ψ relative to the horizontal, as depicted in figure 1. The equations for u, ψ, θ are

$$\dot{u} = -\frac{\rho\sigma u^2}{2m}c_d(\alpha), \tag{12}$$

$$\dot{\psi} = \frac{\rho\sigma u}{2m}c_l(\alpha), \tag{13}$$

$$\ddot{\theta} = \frac{\rho\sigma l u^2}{2I}c_m(\alpha). \tag{14}$$

Here, α is related to θ and ψ by

$$\theta = \psi + \alpha, \tag{15}$$

which we see by inspection of figure 6.

A simple approximate reduction informs the dynamics of the ya's alignment with its direction of motion. First, we ignore the decay of the speed u. With u taken to be constant, the prefactors in (13) and (14) are likewise constant, independent of time. Next, we evoke the linearizations of $c_l(\alpha)$, $c_m(\alpha)$ as in (7), valid for small angle of attack. In summary, the reductions of (13) and (14) are

$$\dot{\psi} \sim A\alpha, \; A := \frac{\rho\sigma u}{2m}a, \tag{16}$$

$$\ddot{\theta} \sim -B\alpha, \; B := \frac{\rho\sigma l u^2}{2I}b. \tag{17}$$

From (15), (16), (17) we find that the dynamics of α is

$$\ddot{\alpha} + A\dot{\alpha} + B\alpha = 0. \tag{18}$$

If $B > 0$ (center of lift behind center of mass), (18) represents *damped harmonic oscillations*. A physical picture of the dynamics: The lift force concentrated behind the center of mass induces a restoring torque that aligns the arrow with its direction of motion. Simultaneously, it changes the direction of motion as well, and this the origin of the damping $A\dot{\alpha}$ in (18).

As we shall see, the oscillator (18) is *underdamped* with $B > \left(\frac{A}{2}\right)^2$ for typical fletched arrows. In this case, the solutions for α are

$$\alpha = e^{-\frac{A}{2}t}\cos\omega t, \; \omega := \sqrt{B - \left(\frac{A}{2}\right)^2}, \tag{19}$$

modulo a multiplicative constant, and translation of the origin of time. From the information in the caption of figure 7, we can work out the values of constants A and B for the test setup:

$$A \approx 1.02 \; sec^{-1}, \; B \approx 307 \; sec^{-2}. \tag{20}$$

The oscillation frequency in Hertz is

$$\nu = \frac{\omega}{2\pi} \approx 2.79 \; sec^{-1}. \tag{21}$$

The e-folding time for the decay of the oscillations is

$$\frac{2}{A} \approx 1.96 \; sec. \tag{22}$$

As noted in the caption of figure 9, the airspeed of the wind tunnel is about half of the typical arrow speeds in Olympic competition. [1] and [2] haven't reported measurements in their magnetic suspension facility at these higher speeds, so we resort to a rough approximation, in which the lift and pitching coefficients have the *same* dependences upon the angle of attack as before. Since $A \propto u$ and $B \propto u^2$, we find that the frequency ν scales by the ratio of speeds. In a test of shot arrows reported in [3], the initial arrow speed was $56.4 m/sec$, which is 1.8 times faster than the wind tunnel speed. Our estimate of the frequency for the higher speed arrow is $\nu \approx 5.02\ sec^{-1}$. The decay time of the oscillations scales with the reciprocal of the speed ratio, and for the faster arrow it is $\frac{2}{A} \approx 1.09\ sec$. An arrow shot at a target with the range of $60m$ will do about five "porpoising" oscillations, and these oscillations decay to less than half of their initial amplitude by the time the arrow lands.

As the angle of attack oscillates, so does the center of mass elevation. Given the speed u and angle of motion ψ, the vertical component of center of mass velocity $\mathbf{u}$ is

$$w = u \sin \psi,$$

or for small ψ,

$$w \sim u\psi. \tag{23}$$

The center of mass elevation y satisfies

$$\dot{y} = w \sim u\psi. \tag{24}$$

From (23) and (16) we have

$$\ddot{y} \sim uA\alpha. \tag{25}$$

Given α as in (19), the solution for y which decays to zero as $t \to \infty$ is

$$y \sim \frac{uA}{B} e^{-\frac{A}{2}t} \cos(\omega t + \zeta), \tag{26}$$

where the *phase shift* ζ is

$$\zeta = 2 \arctan\left(\frac{2\omega}{A}\right). \tag{27}$$

Here, we've continued treating u as a constant. The pre-factor $\frac{uA}{B}$ carries the unit of length, as it should. It's meaning: If the oscillations in the angle of attack have an amplitude of α_o radians, the elevation oscillates with amplitude

$$y_o = \frac{uA}{B}\alpha_o. \tag{28}$$

Recalling that A is proportional to u and B proportional to u^2, we see that the amplitude of elevation oscillations is independent of arrow speed. For the Olympic arrow tested in the wind tunnel, flying at $31.4m/s$, let's say the "porpoising" oscillation has an amplitude $\alpha_o = 1\ deg \approx .0176\ rad$. From the values of A, B in (20), we find that the elevation oscillates with amplitude

$$y_o \approx \frac{(3140cm\ s^{-1})(1.01s^{-1})}{(307s^{-2})}(.0176)\ cm \approx .177cm. \tag{29}$$

Quite insignificant.

Let's look at the meaning of the phase shift ζ in (27). Porpoising oscillations of typical fletched arrows are underdamped, with $\frac{2\omega}{A} >> 1$. For instance, the test case with A and B as in (20) has $\frac{2\omega}{A} \approx 34.4$. In this case, the phase shift ζ is near π. The porpoising and elevation oscillations are one half cycle out of phase. Figure 6 is a qualitative movie of the ya during one cycle of "porpoising."

References

[1] Yuri B. Chernyak and Robert Rose, "The Chicken from Minsk," Basic Books (1995).

[2] Miyazaki, T.; Mukaiyama, K.; Komori, K.; Okawa, K.; Taguchi, S.; Sugiura, H. Aerodynamic properties of an archery arrow. Sports Eng. 2013, 16, 43-54.

[3] Julio Ortiz, Atsushi Serino, Toshinari Hasegawa, Takahito Onoguchi, Hiroki Maemukai, Takeshi Miyazaki, Hiroki Sugiura. MPDI Proceedings 2020, 49, 56.